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Racah-type expressions for the 6*j* coefficients of the orthosymplectic superalgebra osp(1,2)

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Abstract. Using generating functions, various explicit expressions for the 6*j* coefficients of the osp(1,2) superalgebra are derived. Some of these expressions bear a close resemblance to the Racah formula for the su(2) 6*j* coefficients. As a consequence it is shown that the osp(1,2) 6*j* coefficients exhibit Regge symmetries.

1. Introduction

In this work we establish formulae for the 6*j* coefficients of the osp(1,2) superalgebra, sometimes denoted by B(0,1). The finite-dimensional representations of this superalgebra are labelled by a superspin which reminds us of the su(2) spin label. In fact, several authors have developed the Racah–Wigner calculus for the osp(1,2) superalgebra, showing that many properties of the su(2) Racah–Wigner calculus (Clebsch–Gordan coefficients, 3*j* and 6*j* symbols, tensor operators, Wigner–Eckart theorem, Wigner and Racah operators, Biedenharn–Elliott identity,...) have their counterparts in the osp(1,2) superalgebra (Scheunert *et al* 1977, Berezin and Tolstoy 1981, Zeng 1987a, b, Zeng and Yuan 1988, Minnaert and Mozrzymas 1992a, b).

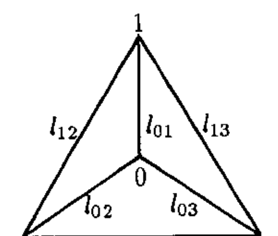

Figure 1. Jucys graph of the 6*j* coefficient.

By coupling and recoupling three superspins, Zeng (1987a) defined Racah (6*j*) coefficients that are invariant but for a possible change of sign in the $4! = 24$ permutations of the vertices of the tetrahedron (figure 1) representing the coefficients. Minnaert and Mozrzymas (1992a) defined more symmetrical 6*j* coefficients which remain completely invariant in these 24 permutations. In this paper we study these latter coefficients, slightly modified by multiplication by an invariant phase factor.

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In section 2, we recall the generating function of su(2) $6j$ coefficients. In section 3, we start from an expression of the osp(1,2) $6j$ coefficients as a sum of eight su(2) $6j$ coefficients and define generating functions, distinguishing eight classes of coefficients. In sections 4–6, for these various classes, the generating functions are calculated and expressions for the coefficients are derived. Some of these expressions bear a close resemblance to the Racah formula for the su(2) $6j$ coefficients, from which it follows that the osp(1,2) $6j$ coefficients present not only the 24 symmetries of the regular tetrahedron but also additional Regge-type symmetries. We also give expressions in terms of the chromatic polynomial, which can be viewed as a terminating ${}_4F_3$ hypergeometric series.

2. The su(2) $6j$ coefficients

We define a number of notations, most of which are adapted from Bargmann (1962), (see also Biedenharn and Van Dam (1965, p 300–16)) and from Labarthe (1975). The su(2) $6j$ coefficient

$$\begin{Bmatrix} l_{01} & l_{02} & l_{03} \\ l_{23} & l_{13} & l_{12} \end{Bmatrix} \tag{2.1}$$

is represented in figure 1 by its Jucys graph (see Jucys and Bandzaitis 1965). For vertex 0, where the three spins l_{01}, l_{02} and l_{03} meet, we define

$$V_0 = l_{01} + l_{02} + l_{03} \qquad L_{01} = V_0 - 2l_{01} \qquad L_{02} = V_0 - 2l_{02} \qquad L_{03} = V_0 - 2l_{03}. \tag{2.2}$$

We call L_{01}, L_{02} and L_{03} the indices of vertex 0. The triangle condition (l_{01}, l_{02}, l_{03}) is equivalent to the condition: $L_{0i} \in \mathbb{N}$ $(1 \leqslant i \leqslant 3)$ where $\mathbb{N}$ is the set of non-negative integers.

Similarly to equation (2.2), V_i and the indices L_{ij} $(i, j = 0, 1, 2, 3;\ i \neq j)$ are defined for the 4 triangle conditions of the $6j$. We put together the indices as $L = (L_{01}, L_{02}, \ldots, L_{32})$. When the spins l_{ij} in array (2.1) take all possible values compatible with triangle conditions, L runs on a subset E of $\mathbb{N}^{12}$ (the indices are not independent: there are six relations like $L_{01} + L_{02} = L_{31} + L_{32} = 2l_{03}$). We also denote by E the set of the corresponding arrays of six spins

$$L = \begin{bmatrix} l_{01} & l_{02} & l_{03} \\ l_{23} & l_{13} & l_{12} \end{bmatrix}$$

(using the same symbols E and L). With this convention, $\{L\}$ denotes the value of the $6j$ coefficient (2.1).

We make the following definitions (see Labarthe 1986). The operations $L + L'$ and μL for $L, L' \in E$ and $\mu \in \mathbb{N}$ are the usual matrix operations. It is easily seen that $L + L' \in E$ and $\mu L \in E$ (that is E is closed under these operations). An element $L \in E$ is called *extremal* if it cannot be decomposed as a sum $L = L' + L''$ of non-zero elements $L', L'' \in E$. There are 7 extremal elements, denoted by e_i $(1 \leqslant i \leqslant 7)$, which are defined in table 1.

Table 1. The extremal elements $e_i = \begin{bmatrix} l_{01} & l_{02} & l_{03} \\ l_{23} & l_{13} & l_{12} \end{bmatrix}$ and the associated monomials $z_i = t^{[e_i]}$ $(1 \leqslant i \leqslant 7)$. Note the useful relation $z_1 z_2 z_3 = z_4 z_5 z_6 z_7$ (see equation (2.8)).

$e_1 = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$	$z_1 = t_{01} t_{10} t_{23} t_{32}$
$e_2 = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$	$z_2 = t_{02} t_{20} t_{13} t_{31}$
$e_3 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$	$z_3 = t_{12} t_{21} t_{03} t_{30}$
$e_4 = \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$	$z_4 = t_{10} t_{20} t_{30}$
$e_5 = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 \end{bmatrix}$	$z_5 = t_{01} t_{21} t_{31}$
$e_6 = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{bmatrix}$	$z_6 = t_{02} t_{12} t_{32}$
$e_7 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$	$z_7 = t_{03} t_{13} t_{23}$

The generating function Φ of the $6j$ is an entire function of $t = (t_{01}, t_{02}, \ldots, t_{32})$ given by

$$\Phi = \sum_{L \in E} N_L \{L\} t^{[L]} = \frac{1}{S^2} \tag{2.3}$$

where N_L is the normalization constant

$$N_L = \left(\prod_{i=0}^{3} \frac{(V_i + 1)!}{\prod_{j \neq i} L_{ij}!} \right)^{1/2} \tag{2.4}$$

and where $t^{[L]} = t_{01}^{L_{01}} t_{02}^{L_{02}} \ldots t_{32}^{L_{32}}$. The generating function Φ is expressed in terms of

$$S = 1 + z_1 + z_2 + z_3 + z_4 + z_5 + z_6 + z_7 \tag{2.5}$$

where the z_i are given in table 1.

The expansion of $1/S^2$ in equation (2.3) gives the value of the $6j$

$$\{L\} = \frac{1}{N_L} \sum_{\alpha} \frac{(-1)^{|\alpha|}(|\alpha| + 1)!}{\alpha_1! \alpha_2! \alpha_3! \alpha_4! \alpha_5! \alpha_6! \alpha_7!} \tag{2.6}$$

where $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_7$ and where the summation is over the $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_7) \in \mathbb{N}^7$ such that

$$L = \sum_{i=1}^{7} \alpha_i e_i . \tag{2.7}$$

Since the extremal elements e_i are linked by

$$e_1 + e_2 + e_3 = e_4 + e_5 + e_6 + e_7 \tag{2.8}$$

the sum in equation (2.6) can be written in terms of a summation on one integer thus recovering Racah's formula for the $6j$.

Let us recall the interpretation of equation (2.6) in the chromatic method of evaluating Penrose spin networks (see Penrose 1979, Moussouris 1979 and Kauffman 1991). For each decomposition of L in $|\alpha|$ extremal elements (2.7), we assign to the extremal elements $|\alpha|$ different colours taken from a set of n colours (n is supposed large enough). The number of ways of decomposing L in these coloured extremal elements is

$$P(L,n) = \sum_{\alpha} \frac{n(n-1)\cdots(n-|\alpha|+1)}{\alpha_1!\alpha_2!\alpha_3!\alpha_4!\alpha_5!\alpha_6!\alpha_7!} \tag{2.9}$$

where the summation is over the α such that equation (2.7) holds. We call $P(L,n)$ the *chromatic polynomial*: for given L, it is defined for any n by equation (2.9) as a polynomial in n that takes integer values when n is a positive or negative integer. The $6j$ coefficient (2.6) and the chromatic polynomial can be written in terms of ${}_4F_3$ hypergeometric functions with unit argument (see Biedenharn and Louck (1981, p 429)). Evaluating the chromatic polynomial at $n = -2$, which corresponds to the 'number of colours' of spin networks, permits to write equation (2.6) as

$$\{L\} = N_L^{-1} P(L,-2). \tag{2.10}$$

The symmetries of the $6j$ (including the Regge (1959) symmetries) are easily obtained from equation (2.6). They correspond to permutations of extremal elements within (e_1, e_2, e_3) and (e_4, e_5, e_6, e_7). These $6 \times 24 = 144$ permutations leave equations (2.4), (2.6) and (2.8) unchanged.

3. The osp(1,2) $6j$ coefficients

The finite-dimensional representations of the superalgebra osp(1,2) are labelled by a superspin j which takes integer or half-integer values $j = 0, \frac{1}{2}, 1, \ldots$ (see Pais and Rittenberg 1975, Scheunert, Nahm and Rittenberg 1977, Hughes 1981 and Berezin and Tolstoy 1981). The representation j has dimension $4j + 1$. When narrowed to the su(2) algebra, it splits into two (or one for $j = 0$) representations of su(2)

$$j \to l = j \qquad \text{and} \qquad l = j - 1/2 \qquad (\text{if } j \neq 0). \tag{3.1}$$

We also write this in the form $l = j - k/2$ with $k = 0$ or 1. These representation spaces are graded, all states in $l = j$ have the same degree denoted by λ (λ can take the values 0 and 1) and all states $l = j - 1/2$ have degree $1 - \lambda$.

The coupling of two superspins j_1 and j_2 yields the values $j_3 = |j_1 - j_2|$, $|j_1 - j_2| + 1/2, \ldots, j_1 + j_2$ of the resultant superspin. The degrees of the coupled states are determined by

$$\lambda_3 \equiv \lambda_1 + \lambda_2 + 2(j_1 + j_2 + j_3) \pmod 2 \tag{3.2}$$

where λ_i specifies the degree in representation space j_i ($i = 1, 2, 3$).

We define the osp(1,2) $6j$ coefficients by

$$\begin{Bmatrix} j_{01} & j_{02} & j_{03} \\ j_{23} & j_{13} & j_{12} \end{Bmatrix}_s = \sum (-1)^{\phi} F_0 F_1 F_2 F_3 \begin{Bmatrix} l_{01} & l_{02} & l_{03} \\ l_{23} & l_{13} & l_{12} \end{Bmatrix} \tag{3.3}$$

where

$$\begin{aligned} j_{mn} &= l_{mn} + k_{mn}/2 \qquad (k_{mn} = 0 \text{ or } 1) \qquad (0 \leqslant m < n \leqslant 3) \\ \phi &= |k| + k_{01}k_{23} + k_{02}k_{13} + k_{03}k_{12} \\ |k| &= k_{01} + k_{02} + k_{03} + k_{23} + k_{13} + k_{12}. \end{aligned} \tag{3.4}$$

The sum in equation (3.3) is over the su(2) spins l_{mn} that correspond to the splitting (3.1) of the osp(1,2) spins j_{mn}. In equation (3.3), for each vertex i in figure 1 there appears a *vertex factor* F_i which depends on the j_{mn} and l_{mn} (with m or $n = i$) that meet at vertex i. In table 2, we give the values of F_0, the other F_i being defined similarly.

Table 2. The vertex factors F_0 and f_0 at vertex 0 for the eight possible values of $k_{01} = 2(j_{01} - l_{01})$, $k_{02} = 2(j_{02} - l_{02})$ and $k_{03} = 2(j_{03} - l_{03})$. Notice that the various vertex factors F_0 are obtained one from the other, but for a phase, by effecting for each change of k_{0i} the mirror transformation $l_{0i} \to -l_{0i} - 1$.

k_{01}	k_{02}	k_{03}	F_0	f_0
0	0	0	$(l_{01} + l_{02} + l_{03} + 1)^{1/2}$	1
0	0	1	$(l_{01} + l_{02} - l_{03})^{1/2}$	L_{03}
0	1	0	$(l_{03} + l_{01} - l_{02})^{1/2}$	L_{02}
0	1	1	$-(l_{02} + l_{03} - l_{01} + 1)^{1/2}$	-1
1	0	0	$(l_{02} + l_{03} - l_{01})^{1/2}$	L_{01}
1	0	1	$-(l_{03} + l_{01} - l_{02} + 1)^{1/2}$	-1
1	1	0	$-(l_{01} + l_{02} - l_{03} + 1)^{1/2}$	-1
1	1	1	$(l_{01} + l_{02} + l_{03} + 2)^{1/2}$	$L_{01} + L_{02} + L_{03} + 2$

Definition (3.3) is independent of the λ_{mn} that specify the degrees in the spaces j_{mn}. It has been arrived at by removing the dependency on λ_{mn} from the osp(1,2) $6j$ coefficients defined by Minnaert and Mozrzymas (1992a). In the appendix, we give the Racah ($6j$) coefficients defined by Zeng (1987a) and Minnaert and Mozrzymas (1992a) in terms of the $6j$ coefficients (3.3).

As in equation (2.2), for the three coupled superspins at vertex 0, j_{01}, j_{02} and j_{03}, we define

$$W_0 = j_{01} + j_{02} + j_{03} \qquad J_{01} = W_0 - 2j_{01} \qquad J_{02} = W_0 - 2j_{02} \qquad J_{03} = W_0 - 2j_{03} \tag{3.5}$$

and analogous quantities W_m, J_{mn} at the other three vertices $m = 1, 2, 3$. The supertriangle condition (j_{01}, j_{02}, j_{03}) is now equivalent to the condition that the indices J_{0i} at vertex 0 are non-negative integers or half-integers.

When the superspins j_{mn} in the $6j$ take all possible values compatible with supertriangle conditions, $J = (J_{01}, J_{02}, \ldots, J_{32})$ runs on a set E_s that contains the set E of possible su(2) indices. As in section 2, we also use symbol J to represent the array of six spins j_{mn}, for example in $\{J\}_s$ to denote the value of a $6j$ coefficient. The set E_s, also identified with the set of six spin arrays that satisfy the supertriangle conditions of the $6j$, is closed under addition and under multiplication by a non-negative integer. The e_i ($1 \leqslant i \leqslant 7$) of table 1 are still extremal elements in E_s, but there are now 13 other extremal elements g_i, $\bar{g}_i$ ($1 \leqslant i \leqslant 6$) and g_7. The elements g_3, $\bar{g}_3$ and g_7 are given in table 3. The other g_i, $\bar{g}_i$ differ from g_3, $\bar{g}_3$ by exchanging spin j_{03} (its values are 0 or 1) with another one (j_{01} for g_1 and $\bar{g}_1, \ldots$).

Table 3. The extremal elements g_3, $\bar{g}_3$ and g_7, defined as arrays $\begin{bmatrix} j_{01} & j_{02} & j_{03} \\ j_{23} & j_{13} & j_{12} \end{bmatrix}$ and in terms of the e_i (table 1). The other g_i, $\bar{g}_i$ ($1 \leqslant i \leqslant 6$) differ from g_3, $\bar{g}_3$ by exchanging spin j_{03} (its values are 0 or 1) with another one (j_{01} for g_1 and $\bar{g}_1, \ldots$).

$$g_3 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \frac{e_3 + e_4 + e_7}{2}$$

$$\bar{g}_3 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \frac{e_1 + e_2 + e_5 + e_6}{2}$$

$$g_7 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \frac{e_1 + e_2 + e_3}{2} = \frac{e_4 + e_5 + e_6 + e_7}{2}$$

The osp(1,2) indices $J \in E_s$ can be classified in eight classes as follows (in other words the set E_s is the union of eight disjoint subsets E_i ($0 \leqslant i \leqslant 7$)): $E_0 = E$ is the class of indices that are also su(2) indices; E_i ($1 \leqslant i \leqslant 6$) is the class of indices of the form $g_i + u$ or $\bar{g}_i + u$ with $u \in E$; E_7 is the class of indices of the form $g_7 + u$ with $u \in E$. This classification is the same (in different order) as in Zeng (1987a) where the parities of $2W_m$ ($0 \leqslant m \leqslant 3$) are used to characterize the different classes.

For the osp(1,2) $6j$ coefficients, we define a normalization constant

$$M_J = \left(\prod_{i=0}^{3} \frac{\lfloor W_i + 1/2 \rfloor !}{\prod_{j \neq i} \lfloor J_{ij} \rfloor !} \right)^{1/2} \tag{3.6}$$

where $\lfloor x \rfloor$ designates the greatest integer smaller than or equal to x. With this definition we rewrite equation (3.3) as

$$M_J \{J\}_s = \sum (-1)^{\phi} f_0 f_1 f_2 f_3 N_L \{L\} \tag{3.7}$$

where the vertex factors f_i take now integer values (table 2 gives the vertex factor f_0 for vertex 0). In order to obtain expressions for the $6j$ coefficients it is convenient to consider the classes E_i ($0 \leqslant i \leqslant 7$) separately. So, using similar notations as for equation (2.3), we define the generating function Ψ_i ($0 \leqslant i \leqslant 7$) of the osp(1,2) $6j$ coefficients with indices in E_i as the entire function of $t_{mn}^{1/2}$ given by

$$\Psi_i = \sum_{J \in E_i} M_J \{J\}_s t^{[J]} = \sum_{L \in E} \left(\sum_{K} (-1)^{\phi} f_0 f_1 f_2 f_3 \, t^{[K/2]} \right) N_L \{L\} t^{[L]} \tag{3.8}$$

where $K = 2(J - L)$ are the indices that correspond to the array

$$\begin{bmatrix} k_{01} & k_{02} & k_{03} \\ k_{23} & k_{13} & k_{12} \end{bmatrix}.$$

The sum over K in equation (3.8) is limited to the values of this array with $k_{mn} = 0$ or 1 and such that there exists $L \in E$ for which $L + K/2 \in E_i$.

4. Calculation of $\{J\}_s$: case $J \in E$

When $J \in E$, the K in equation (3.8) takes the eight values such that $K/2 \in E$. These are in fact $K = 0$ and $K = 2e_i$ $(1 \leqslant i \leqslant 7)$ (see table 1). The factor $(-1)^{\phi} f_0 f_1 f_2 f_3 t^{[K/2]}$ has value 1 for $K = 0$ and z_i (see table 1) for $K = 2e_i$ $(1 \leqslant i \leqslant 7)$. Using equations (2.3) and (2.5) we obtain

$$\Psi_0 = \sum_{J \in E} M_J \{J\}_s t^{[J]} = \frac{1}{S}. \tag{4.1}$$

By expanding $1/S$ in equation (4.1) we derive an expression similar to equation (2.6) for the value of the $6j$

$$\{J\}_s = \frac{1}{M_J} \sum_{\alpha} \frac{(-1)^{|\alpha|} |\alpha|!}{\alpha_1! \alpha_2! \alpha_3! \alpha_4! \alpha_5! \alpha_6! \alpha_7!} \tag{4.2}$$

where the summation is over the $\alpha \in \mathbb{N}^7$ such that $J = \sum_{i=1}^{7} \alpha_i e_i$.

Solving $J = \sum_{i=1}^{7} \alpha_i e_i$ for the α_i $(1 \leqslant i \leqslant 7)$ in terms of j_{mn} and $|\alpha| = x$

$$\begin{aligned} \alpha_1 &= j_{02} + j_{03} + j_{13} + j_{12} - x & \alpha_5 &= x - j_{01} - j_{13} - j_{12} \\ \alpha_2 &= j_{01} + j_{03} + j_{23} + j_{12} - x & \alpha_6 &= x - j_{23} - j_{02} - j_{12} \\ \alpha_3 &= j_{01} + j_{02} + j_{23} + j_{13} - x & \alpha_7 &= x - j_{23} - j_{13} - j_{03} \\ \alpha_4 &= x - j_{01} - j_{02} - j_{03} \end{aligned} \tag{4.3}$$

shows explicitly that the summation in equation (4.2) is over integer x such that the α_i $(1 \leqslant i \leqslant 7)$ are non-negative integers.

The value of the $6j$ coefficient can also be written in terms of the chromatic polynomial (2.9) evaluated at $n = -1$

$$\{J\}_s = M_J^{-1} P(J, -1). \tag{4.4}$$

From equation (4.2) we obtain that the osp(1,2) $6j$ symbol $\{J\}_s$ for $J \in E$ presents the same 144 Regge symmetries as the su(2) $6j$ symbol.

Finally, let us mention that equation (4.1), written in the form $\Psi_0 S = 1$, gives the summation formula

$$\sum_{J'} M_{J'} \{J'\}_s = \delta_{J,0} \tag{4.5}$$

where J is fixed and where the summation is over the eight values $J' = J$, $J' = J - e_i$ $(1 \leqslant i \leqslant 7)$. The delta function on the right has value 0 except for $J = 0$ for which it takes the value 1.

5. Calculation of $\{J\}_s$: case $J \in E_3$

All cases E_i ($1 \leqslant i \leqslant 6$) are similar. We treat the case $J \in E_3$. The K in equation (3.8) takes eight values which are given in table 4 by the corresponding values of k_{mn}. The factor $(-1)^\phi f_0 f_1 f_2 f_3$ now depends on the indices L_{mn}. So, for the first value of K in table 4, we have $(-1)^\phi f_0 f_1 f_2 f_3 = -L_{21} L_{12}$. Replacing the indices L_{mn} by operators $\hat{L}_{mn}$, for instance

$$L_{12} \to \hat{L}_{12} = t_{12} \frac{\partial}{\partial t_{12}} = z_3 \frac{\partial}{\partial z_3} + z_6 \frac{\partial}{\partial z_6} \tag{5.1}$$

we can carry out the summation over L in equation (3.8)

$$\Psi_3 = \sum_K R_K \left(\sum_{L \in E} N_L \{L\} t^{[L]} \right) = \sum_K R_K \Phi \tag{5.2}$$

where the operators R_K are detailed in table 4 and Φ is the generating function (2.3) of the su(2) $6j$ coefficients. Applying the operators R_K to $\Phi = 1/S^2$, we can express Ψ_3 in terms of the z_i as

$$\Psi_3 = \sum_{J \in E_3} M_J \{J\}_s t^{[J]} = 2 \frac{(z_3 z_4 z_7)^{1/2} - (z_1 z_2 z_5 z_6)^{1/2}}{S^3}. \tag{5.3}$$

Relations (2.5) and $z_1 z_2 z_3 = z_4 z_5 z_6 z_7$ have been used to derive this equation.

Table 4. Calculation of equation (5.2). The K are given by their k_{mn} values. The relation $(z_3 z_4 z_7)^{1/2} = (z_1 z_2 z_5 z_6)^{1/2} z_3 / (z_5 z_6)$ is useful when applying the operator R_K to Φ.

k_{01}	k_{02}	k_{03}	k_{23}	k_{13}	k_{12}	R_K
0	0	0	0	0	1	$-(z_3 z_4 z_7)^{1/2} \frac{1}{z_3} \hat{L}_{21} \hat{L}_{12}$
0	1	1	0	1	0	$(z_1 z_2 z_5 z_6)^{1/2} \frac{1}{z_2} \hat{L}_{20} \hat{L}_{13}$
1	0	1	1	0	0	$(z_1 z_2 z_5 z_6)^{1/2} \frac{1}{z_1} \hat{L}_{10} \hat{L}_{23}$
1	1	0	1	1	1	$-(z_3 z_4 z_7)^{1/2} (\hat{L}_{10} + \hat{L}_{12} + \hat{L}_{13} + 2)(\hat{L}_{20} + \hat{L}_{21} + \hat{L}_{23} + 2)$
0	0	0	1	1	0	$-(z_3 z_4 z_7)^{1/2} \frac{1}{z_7} \hat{L}_{13} \hat{L}_{23}$
0	1	1	1	0	1	$-(z_1 z_2 z_5 z_6)^{1/2} \frac{1}{z_6} \hat{L}_{12} (\hat{L}_{20} + \hat{L}_{21} + \hat{L}_{23} + 2)$
1	0	1	0	1	1	$-(z_1 z_2 z_5 z_6)^{1/2} \frac{1}{z_5} \hat{L}_{21} (\hat{L}_{10} + \hat{L}_{12} + \hat{L}_{13} + 2)$
1	1	0	0	0	0	$-(z_3 z_4 z_7)^{1/2} \frac{1}{z_4} \hat{L}_{10} \hat{L}_{20}$

We obtain a formula for the $6j$ coefficient $\{J\}_s$ ($J \in E_3$) by expanding S^{-3} in equation (5.3). The coefficient $\{J\}_s$ is expressed as a sum over the decompositions of J in the forms

$$J = g_3 + \sum_{i=1}^{7} \alpha_i e_i \qquad \text{and} \qquad J = \bar{g}_3 + \sum_{i=1}^{7} \bar{\alpha}_i e_i \tag{5.4}$$

where $\alpha, \bar{\alpha} \in \mathbb{N}^7$

$$\{J\}_s = \frac{1}{M_J}\left(\sum_\alpha \frac{(-1)^{|\alpha|}(|\alpha|+2)!}{\alpha_1!\alpha_2!\alpha_3!\alpha_4!\alpha_5!\alpha_6!\alpha_7!} - \sum_{\bar{\alpha}} \frac{(-1)^{|\bar{\alpha}|}(|\bar{\alpha}|+2)!}{\bar{\alpha}_1!\bar{\alpha}_2!\bar{\alpha}_3!\bar{\alpha}_4!\bar{\alpha}_5!\bar{\alpha}_6!\bar{\alpha}_7!}\right). \tag{5.5}$$

This gives an expression in terms of 2 chromatic polynomials evaluated at $n = -3$

$$\{J\}_s = 2M_J^{-1}(P(J - g_3, -3) - P(J - \bar{g}_3, -3)). \tag{5.6}$$

We can recast equation (5.5) as a sum over the decompositions of J in the form

$$J = \sum_{i=1}^{7} \beta_i \frac{e_i}{2} \tag{5.7}$$

where $\beta \in \mathbb{N}^7$

$$\{J\}_s = \frac{1}{M_J}\sum_\beta \frac{(-1)^{\lfloor(|\beta|+1)/2\rfloor+|\beta|+1}\lfloor(|\beta|+1)/2\rfloor!}{\lfloor\beta_1/2\rfloor!\lfloor\beta_2/2\rfloor!\lfloor\beta_3/2\rfloor!\lfloor\beta_4/2\rfloor!\lfloor\beta_5/2\rfloor!\lfloor\beta_6/2\rfloor!\lfloor\beta_7/2\rfloor!}. \tag{5.8}$$

The decompositions (5.4) correspond to the values $\beta = 2\alpha + (0,0,1,1,0,0,1)$ and $\beta = 2\bar{\alpha} + (1,1,0,0,1,1,0)$ in decomposition (5.7).

Although equation (5.8) is quite similar to equation (2.6), we do not obtain all 144 Regge symmetries. Indeed, when we permute extremal elements within (e_1, e_2, e_3) and (e_4, e_5, e_6, e_7), for given β, it can happen that the sum in equation (5.7) is no more in E_s (giving $j_{01} = \frac{1}{4}$ for example). The permutations that leave J in E_s can be obtained by combining the $2 \times 2 \times 2$ permutations that leave the three pairs (e_1, e_2), (e_4, e_7) and (e_5, e_6) unchanged (so that g_3 and $\bar{g}_3$ also remain unchanged) with six permutations that change g_3 into g_i ($1 \leqslant i \leqslant 6$). So, there are in fact $8 \times 6 = 48$ symmetries. They include of course the 24 permutations of the vertices of the tetrahedron in figure 1. One example of the remaining symmetries is given by the exchange $e_1 \leftrightarrow e_2$

$$\begin{Bmatrix} j_{01} & j_{02} & j_{03} \\ j_{23} & j_{13} & j_{12} \end{Bmatrix}_s = \begin{Bmatrix} c - j_{23} & c - j_{13} & j_{03} \\ c - j_{01} & c - j_{02} & j_{12} \end{Bmatrix}_s \qquad c = \frac{j_{01} + j_{02} + j_{13} + j_{23}}{2}. \tag{5.9}$$

Noting that for

$$J = \begin{bmatrix} j_{01} & j_{02} & j_{03} \\ j_{23} & j_{13} & j_{12} \end{bmatrix} \in E_3$$

we have

$$J_0 = J - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \in E$$

we may wonder whether $\{J\}_s$ can be expressed in terms of the chromatic polynomial $P(J_0, n)$. That this can be done we show as follows. Writing equation (5.3) as

$$\Psi_3 = 2\left(\frac{z_4 z_7}{z_3}\right)^{1/2} \frac{z_3 - z_5 z_6}{S^3} \tag{5.10}$$

we get be expanding S^{-3}

$$\{J\}_s = \frac{1}{M_J}\sum_\alpha \frac{(-1)^{|\alpha|}|\alpha|!w(\alpha)}{\alpha_1!\alpha_2!\alpha_3!\alpha_4!\alpha_5!\alpha_6!\alpha_7!} \tag{5.11}$$

where $w(\alpha) = -(|\alpha|+1)\alpha_3 - \alpha_5\alpha_6$ and where the sum is over the decompositions of J_0 in the form

$$J_0 = \sum_{i=1}^{7} \alpha_i e_i. \tag{5.12}$$

By solving equation (5.12) for α in terms of J and $|\alpha|$, in a similar way as in equation (4.3), we can rewrite $w(\alpha)$ in the form

$$w(\alpha) = 2j_{12}(|\alpha|+1) - A_J \tag{5.13}$$

where

$$A_J = (W_2+1/2)(W_1+1/2) = (j_{02}+j_{23}+j_{12}+1/2)(j_{01}+j_{13}+j_{12}+1/2). \tag{5.14}$$

Substituting equation (5.13) in equation (5.11), we arrive at an expression in terms of the chromatic polynomial

$$\{J\}_s = M_J^{-1}(2j_{12}P(J_0,-2) - A_JP(J_0,-1)). \tag{5.15}$$

6. Calculation of $\{J\}_s$: case $J \in E_7$

We evaluate equation (3.8) when $J \in E_7$ in the same way as in the preceding section. K runs on a set of eight values as before, but now the factors $(-1)^\phi f_0f_1f_2f_3$ are of degree 4 in the L_{mn}. The generating function Ψ_7 can be arranged as

$$\begin{aligned}\Psi_7 = 6\frac{(z_1z_2z_3)^{1/2}}{S^5}&(1+z_1^2+z_2^2+z_3^2-2(z_1+z_2+z_3+z_1z_2+z_1z_3+z_2z_3)\\ &-(z_4^2+z_5^2+z_6^2+z_7^2)\\ &+2(z_4z_5+z_4z_6+z_4z_7+z_5z_6+z_5z_7+z_6z_7))\end{aligned} \tag{6.1}$$

which is a symmetric function in (z_1, z_2, z_3) and (z_4, z_5, z_6, z_7). It can be remarked that, rather mysteriously, the large polynomial in equation (6.1) looks like the expansion of S^2, except that some signs have changed and that some cross products have vanished.

By expanding S^{-5} in equation (6.1) we obtain a formula for the $6j$ coefficient $\{J\}_s$ ($J \in E_7$) as a sum over the decompositions of J in the form

$$J = g_7 + \sum_{i=1}^{7} \alpha_i e_i \tag{6.2}$$

where $\alpha \in \mathbb{N}^7$

$$\{J\}_s = M_J^{-1} \sum_{\alpha} \frac{(-1)^{|\alpha|}(|\alpha|+2)!w(\alpha)}{\alpha_1!\alpha_2!\alpha_3!\alpha_4!\alpha_5!\alpha_6!\alpha_7!}. \tag{6.3}$$

The factor $w(\alpha)$ which is symmetric in the permutations of $(\alpha_1, \alpha_2, \alpha_3)$ and $(\alpha_4, \alpha_5, \alpha_6, \alpha_7)$ is given by

$$w(\alpha) = 3(1 + \alpha_1 + \alpha_2 + \alpha_3) + 2(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7) + \alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \sum_{1 \leqslant i < j \leqslant 7} \alpha_i \alpha_j. \tag{6.4}$$

We can express equation (6.3) in terms of the chromatic polynomial $P(J - g_7, n)$ in the same way as at the end of section 5. By solving equation (6.2) for α in terms of J and $|\alpha|$, we rewrite $w(\alpha)$ in the form

$$w(\alpha) = -6\frac{|\alpha| + 3}{3!} + \frac{A_J}{2} \tag{6.5}$$

where, for J fixed, A_J does not depend on any peculiar decomposition (6.2). Putting

$$\begin{aligned} \sigma &= j_{01} + j_{02} + j_{03} + j_{23} + j_{13} + j_{12} \\ \tau &= j_{01}j_{23} + j_{02}j_{13} + j_{03}j_{12} \end{aligned} \tag{6.6}$$

we have

$$A_J = 4\tau + 2\sigma + 3. \tag{6.7}$$

Substituting equation (6.5) in equation (6.3), we see that for $J \in E_7$ the $6j$ coefficient $\{J\}_s$ can be expressed in terms of the chromatic polynomial as

$$\{J\}_s = M_J^{-1}(-6P(J - g_7, -4) + A_J P(J - g_7, -3)). \tag{6.8}$$

From equation (6.3) we easily obtain that the osp(1,2) $6j$ symbol $\{J\}_s$ for $J \in E_7$ possesses the same 144 Regge symmetries as the su(2) $6j$ symbol.

7. Conclusion

In conclusion, let us point out a subject where the explicit formulae obtained in this work might be useful. The subject is the study of non-trivial zeros of the $6j$ by means of Diophantine equations based on explicit formulae, as was done in the case of su(2) $6j$ coefficients (see for example Beyer, Louck and Stein 1986, Labarthe 1987 and Srinivasa Rao and Chiu 1989). Examples of zeros for $J \in E_3$ are given by

$$\{J\}_s = \begin{Bmatrix} a & j & j \\ j & b & a + b - 1/2 \end{Bmatrix}_s = 0 \tag{7.1}$$

and

$$\left\{ \begin{matrix} (a+b)/2 & j-(b-a)/2 & j \\ j+(b-a)/2 & (a+b)/2 & a+b-1/2 \end{matrix} \right\}_s = 0 \tag{7.2}$$

where a and b are integers and j is integer or half-integer. These two $6j$ are related by Regge symmetry (5.9). The vanishing of these coefficients follows from equation (5.6) and $P(J-g_3,-3) = P(J-\bar{g}_3,-3)$ that is readily verified since there is only one term in the summation of both chromatic polynomials. The vanishing of

$$\left\{ \begin{matrix} 1 & j & j \\ j & 1 & 3/2 \end{matrix} \right\}_s$$

(corresponding to $a = b = 1$) for any j has been established by Minnaert and Mozrzymas (1992b) by considering a chain of superalgebras.

Appendix

The osp(1,2) $6j$ coefficients defined by Minnaert and Mozrzymas (1992a) are related to the $6j$ coefficients (3.3) by

$$\left\{ \begin{matrix} j_{01} & j_{02} & j_{03} \\ j_{23} & j_{13} & j_{12} \end{matrix} \right\}^{SR} = (-1)^{\theta} \left\{ \begin{matrix} j_{01} & j_{02} & j_{03} \\ j_{23} & j_{13} & j_{12} \end{matrix} \right\}_s \tag{A.1}$$

with

$$\theta \equiv 2W_1\lambda_{23} + 2W_2\lambda_{13} + 2W_3\lambda_{12} \equiv \lambda_{01}\lambda_{23} + \lambda_{02}\lambda_{13} + \lambda_{03}\lambda_{12} \pmod 2 \tag{A.2}$$

where W_m is defined as in equation (3.5). Equation (A.1) follows from equation (39) of Minnaert and Mozrzymas (1992a) by summing over all M and using their equations (24), (38) and (A13). The equivalence of the two forms of θ given in equation (A.2) results from relations like equation (3.2) which occurs for each coupling in the $6j$.

The Racah coefficients defined by Zeng (1987a) in his equations (4.9–4.16) are related to the $6j$ coefficients (3.3) by

$$R(j_{01}, j_{02}, j_{13}, j_{23}; j_{03}, j_{12}) = (-1)^{\psi} \left\{ \begin{matrix} j_{01} & j_{02} & j_{03} \\ j_{23} & j_{13} & j_{12} \end{matrix} \right\}_s \tag{A.3}$$

where

$$\psi = \lfloor j_{01} + j_{02} + j_{13} + j_{23} \rfloor + 4W_2W_3 + 2W_1(1+\lambda_{02}) + 2W_0\lambda_{02}. \tag{A.4}$$

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